ω_1 CAN BE MEASURABLE

BY

T. JECH

ABSTRACT

It is shown that if ZF + the axiom of choice + "there is a measurable cardinal" is consistent then ZF + "ω_1 is measurable" is consistent. The corresponding model is a symmetric submodel of the Cohen-type extension which collapses the first measurable cardinal onto ω_0.

1. **Introduction.** A cardinal number κ is called *measurable* if $\kappa > \omega$ and if there is a κ-complete non-principial ultra filter D on the field of all subsets of κ. It is known in ZF^*, the axiomatic set theory with the axiom of choice, that the least measurable cardinal, if there is any, is very large; bigger than the first inaccessible, bigger than the first Mahlo number etc. (cf. [3]). The present paper shows that without assuming the axiom of choice, it is consistent that ω_1 is measurable. The consistency is relative to ZF^* + "there is a measurable cardinal".

The paper uses methods closely related to those of Lévy and Solovay in [4]. They prove that measurability of κ is preserved under a Cohen-type extension if the set of conditions (or the corresponding complete Boolean algebra) has power less than κ. Here we show that the same is true if one constructs a symmetric extension and if all uniformly symmetric parts of the complete Boolean algebra in question have power less than κ. This is used for construction of a symmetric model where κ becomes ω_1 and remains measurable.

2. **Symmetric extensions.** The reader is assumed to be familiar with Cohen's forcing method [1] and also with the Boolean algebraic approach (see Scott-Solovay [5] or Vopěnka [6]).

Consider a complete Boolean algebra B. By V^B we denote the *Boolean universe*; the real universe V can be embedded into V^B in a usual way; the counterpart of x is denoted by $\check{x}$. For each formula $\phi(v_1, \cdots, v_n)$, $[\![\phi(x_1, \cdots, x_n)]\!]$ denotes the Boolean value of ϕ at $x_1, \cdots, x_n \in V^B$; we say that $\phi(a_1, \cdots, a_n)$ *holds in the extension* if $[\![\phi(\check{a}_1, \cdots, \check{a}_n)]\!] = 1$.

If g is an *automorphism* of B (i.e. g is one-to-one and $x \leqq y \leftrightarrow g(x) \leqq g(y)$) then g can be extended into an automorphism of V^B. A filter F of groups of automorphisms of B is *normal*, if it contains the group $\{g : (\forall G \in F)\, [gGg^{-1} \in F]\}$.

Received June 25, 1968.

Given a normal filter F, an element of V^B is *symmetric* if it is preserved under all automorphisms from some group in F. The class $V^{B,F}$ of all *hereditarily symmeric* elements of V^B is the universe of a symmetric extension (cf. [2]). In a symmetric extension, the axiom of choice does not hold in general.

We now prove that measurability is preserved under certain symmetric extensions. A subset A of B is *uniformly symmetric* if there is some group G in F such that all elements of A are preserved under all automorphisms in G.

THEOREM 1. *Let be a measurable cardinal and D a non-principal κ-complete ultrafilter on κ. Suppose that each uniformly symmetric subset of B has power less than κ. Then it is true in the extension that $\check{D}$ generates a non-principal $\check{\kappa}$-complete ultrafilter $E = \{x \subseteq \check{\kappa} : (\exists y \in \check{D})[x \supseteq y]\}$ on $\check{\kappa}$ and that $\check{\kappa}$ is a cardinal.*

Proof. We prove that the following is true in the extension: If A is a set of subsets of $\check{\kappa}$, if $\bar{\bar{A}} < \check{\kappa}$ and if $\cup A = \check{\kappa}$, then there is some $x \in A$ such that $x \in E$. Every subset of $\check{\kappa}$ in the extension is a symmetric Boolean-valued function on $\{\check{\xi} : \xi < \kappa\}$. The range of such a function is uniformly symmetric and hence has power less than κ. Similarly, if

$$A = \{A_\eta : \eta < \check{\alpha}\}, \qquad \alpha < \kappa$$

and if $\cup A = \check{\kappa}$ in the extension then there exists a Boolean-valued function f on $\kappa \times \alpha$ which has less than κ values and has the following property:

$$\prod_{\xi < \kappa} \sum_{\eta < \alpha} f(\xi, \eta) = 1$$

(This asserts that

$$(\forall \xi < \check{\kappa})(\exists \eta < \check{\alpha})(\xi \in A_\eta).)$$

Therefore we have

$$\sum_{\eta < \alpha} f(\xi, \eta) = 1$$

for all $\xi < \kappa$. Since f has less than κ values and since D is κ-complete there is a $y \in D$ such that $f(\xi_1, \eta) = f(\xi_2, \eta)$ whenever $\xi_1, \xi_2 \in y$ and $\eta < \alpha$. It follows that

$$\sum_{\eta < \alpha} \prod_{\xi \in y} f(\xi, \eta) = 1;$$

hence we obtain

$$(\exists y \in \check{D})(\exists \eta < \check{\alpha})[y \subseteq A_\eta],$$

i.e.,

$$(\exists \eta < \check{\alpha})[A_\eta \in E].$$

A similar argument shows that $\check{\kappa}$ is not collapsed, i.e. that $\check{\kappa}$ is a cardinal in $V^{B,F}$.

In particular constructions of Cohen-type extensions one can use an arbitrary partially ordered set instead of a complete Boolean algebra. It follows however from the following known theorem that we can restrict ourselves, without loss of generality, to complete Boolean algebras. An element of the partially ordered set P is called a *condition.* A condition p is *stronger* than q if $p \leqq q$. Two conditions are *compatible* if there is some condition which is stronger than both of them.

THEOREM 2. *If P is a partially ordered set then there is a unique (up to isomorphism) complete Boolean algebra B and a mapping e of P into $B - \{0\}$ such that*

(i) *$e''P$ is dense in B, i.e. $(\forall x \in B - \{0\})(\exists y \in P)[e(y) \leqq x]$*

(ii) *p, q are compatible in P if and only if $e(p) \cdot e(q) \neq 0$.*

Proof. If P is endowed with the topology given by the base $\{U_p : p \in P\}$ where $U_p = \{q \in P : q \leqq p\}$ then B is the algebra of all regular open sets.

It is also known that if P is used for a construction of a Cohen-type extension and if forcing is defined as in [1] then $p \Vdash \phi \leftrightarrow e(p) \leqq [\![\phi]\!]$ for any closed formula ϕ.

Similarly, we may deal with an arbitrary partially ordered set P of conditions also in case of symmetric extensions An *automorphism* g of P is a one-to-one mapping of P onto P such that $x \leqq y \leftrightarrow g(x) \leqq g(y)$

THEOREM 3. *Let P be a partially ordered set and let B be the complete Boolean algebra mentioned in Theorem 2. If g is an automorphism of P then there is a unique automorphism g of B such that $e(g(p)) = g(e(p))$ for every $p \in P$ If we denote this automorphism by $e(g)$ then we have $(e(g))^{-1} = e(g^{-1})$ and $e(g) \circ e(h) = e(g \circ h)$*

The proof of this theorem is a matter of computation and is based on the fact that the relation $\leqq$ on B is fully defined in terms of the relation $\leqq$ on P and that $e''P$ is dense in the complete Boolean algebra B.

By this theorem, one can construct a symmetric extension using a set of conditions P and a normal filter F of groups of automorphisms of P. (The collection $\{e''G : G \in F\}$ generates a normal filter of groups of automorphisms of B). A set A of sets of conditions is *uniformly symmetric* if there is a group G in F such that for every $g \in G$ and every $S \in A$ we have $g''S = S$. We now may reformulate Theorem 1 as follows:

THEOREM 4. *If κ is a measurable cardinal and if each uniformly symmetric set of sets of conditions has power less than κ then $\check{\kappa}$ is a measurable cardinal in the extension.*

Proof. For $u \in B$ we let $S_u = \{p \in P : e(p) \leqq u\}$. If $A \subseteq B$ is uniformly symmetric then $\{S_u: u \in A\}$ is uniformly symmetric; for, $(e(g))(u) = u$ implies $g''S = S$ for any automorphism g of P. Hence each uniformly symmetric subset of B has power less than κ and we apply Theorem 1.

3. Consistency of ZF + "ω_1 is measurable" relative to ZF^* + "there is a measurable cardinal". First we mention one simple case of symmetric extensions. Let P be a set of conditions and let T be a collection of sets of conditions such that $Z \in T$ implies $g''Z \in T$ for every automorphism g. For every $Z \subseteq P$ denote by G_z the group of all automorphisms g such that $g(p) = p$ whenever $p \in Z$. The filter generated by all groups G_z for $Z \in T$ is normal; hence P and T determine a unique symmetric extension.

We now let P be the set of all functions whose domain is a finite set of integers and whose values are ordinals less than κ; we order P by converse inclusion. (This set of conditions adjoins a collapsing function of $\check{\omega}_0$ onto $\check{\kappa}$, cf. [1].) We let T be the collection of all sets of conditions which have power less than κ. One can prove that every uniformly symmetric set of sets of conditions has power less than κ; this follows from the following

LEMMA. *If A is a uniformly symmetric set of sets of conditions then there is some $\alpha_0 < \kappa$ such that:*

if $S \in A$ and if there is some $p \in S$ and some $n \in \omega$ such that $p(n) > \alpha_0$, then for every $\alpha > \alpha_0$ there is some $q \in S$ such that the domain of q is the domain of p, $q(n) = \alpha$ and $q(m) = p(m)$ for every $m \neq n$.

Consequently, $\overline{\overline{A}} < \kappa$.

Proof. There exists $Z \subseteq P$ such that $\overline{\overline{Z}} < \kappa$ and $g''S = S$ for all $g \in G_Z$ and $S \in A$. We let $\alpha_0 = \sup\{\gamma: \gamma \text{ is in the range of some element of } Z\}$. If $p(n) = \beta > \alpha_0$ and if $\alpha > \alpha_0$ then we let g be the automorphism of P induced by interchanging α and β in all conditions. Clearly, $g \in G_Z$ and $q = g(p)$ has the required property. It follows that the cardinality of A is at most 2^λ where λ is the cardinality of the set of all conditions whose values do not exceed $\alpha_0 + 1$.

By Theorem 4, $\check{\kappa}$ is measurable in the extension. It remains to verify that $\check{\kappa}$ is the first uncountable cardinal in the extension. This follows from the fact that for every $\alpha < \kappa$ there is a symmetric collapsing function of $\check{\omega}_0$ onto $\check{\alpha}$ and that $\check{\kappa}$ is a cardinal in $V^{B,F}$.

REMARK. It is known (cf. [4]) that various properties of cardinals are preserved under "mild" extensions. If analyzing the proofs we find out that many of these properties are preserved also under "mild symmetric" extensions, i.e. extensions satisfying conditions of Theorem 1. E.g., the property of being a weakly compact (strongly compact, Ramsey) cardinal is preserved, so that we have: if ZF^* + "there exists a weakly compact (strongly compact, Ramsey) cardinal" is consistent then ZF + "ω_1 is a weakly compact (strongly compact, Ramsey) cardinal" is consistent.

References

1. P. J. Cohen, *Set theory and the continuum hypothesis*, New York, 1966.

2. T. Jech, *On models without AC*. To appear in the Proceedings of the 1967 Summer Institute on Set Theory in Los Angeles.

3. H. J. Keisler and A Tarski, *From accessible to incessible cardinals*, Fundam. Math., **53**, (1964), 225–308.

4. A. Lévy and R. Solovay, *Measurable cardinals and the continuum hypothesis*, Israel J. of Math., **5** (1967), 234–248.

5. D. Scott and R. Solovay, *Boolean-valued people looking at set theory*. To appear in the Proceedings of the 1967 Summer Institute on Set Theory in Los Angeles.

6. P. Vopěnka, *General theory of* ∇*-models*, Commentat. Math. Univ. Carol., **8** (1967), 145–170.

Department of Mathematics,
Charles University, Prague,
Sokolovska 83, Praha 8, Czechoslovakia